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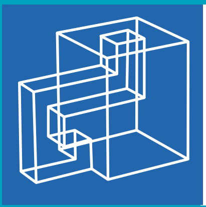


The Klein Bottle and Projective Plane

Skill and Experience ● ○ ○ ○ ○

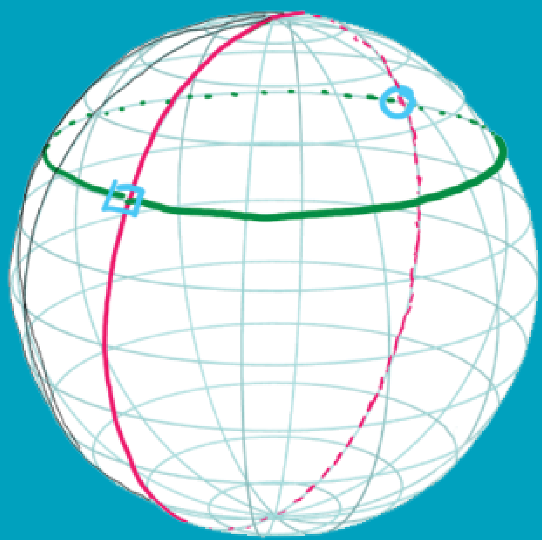
These are easy models to make but need some accuracy and dexterity in the cutting out, folding and sticking.

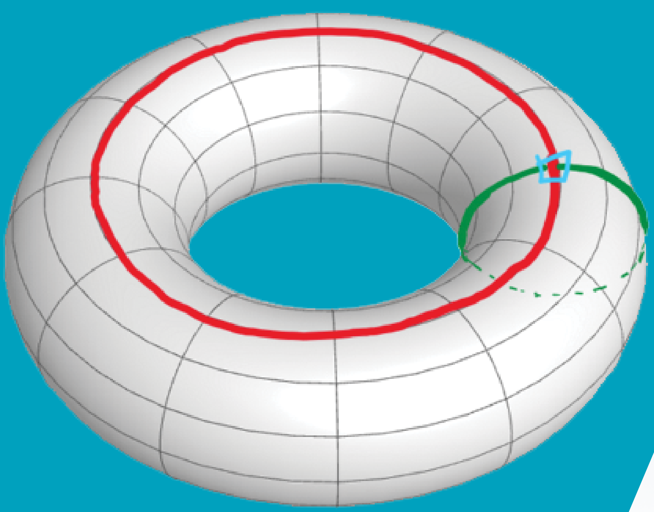
How it relates to maths

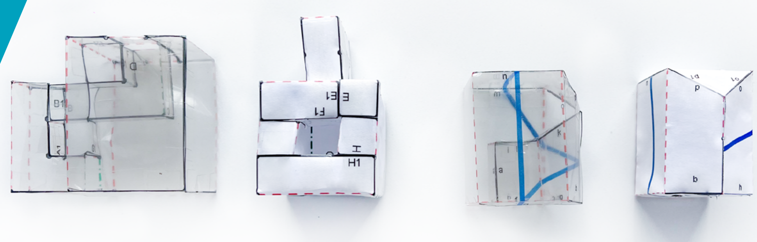
A plane is a two-dimensional surface. Imagine a dot embedded in a horizontal plane. It could move North, South, East or West, but could not move up or down without leaving the plane. If it set off in a straight line it could travel forever.

Now imagine the two-dimensional surface of a sphere. A dot embedded in this surface would have a finite space in which to move, and if it set off in a straight line it would eventually come back to where it started.

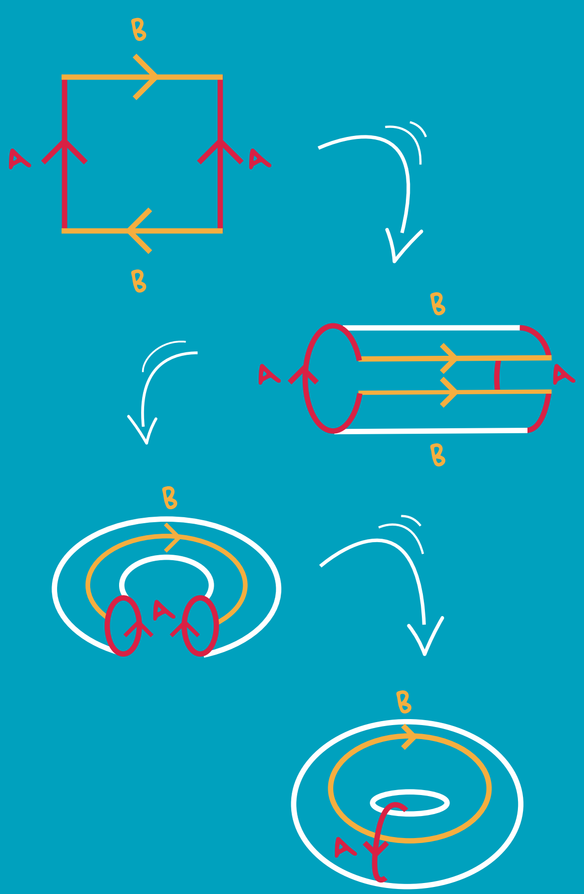
Now imagine the two-dimensional surface of a torus (or ring doughnut). If you were a dot on this surface, how would you know you were not on a sphere? One way would be to walk in a straight line, painting a red line on the ground as you go. When you come back to where the red line starts, turn 90° and set off again painting a green line on the ground as you go. If you cross the red line before you get back to your starting position, you are on a sphere. If you do not cross the red line before you get back, there must be a hole in your surface.



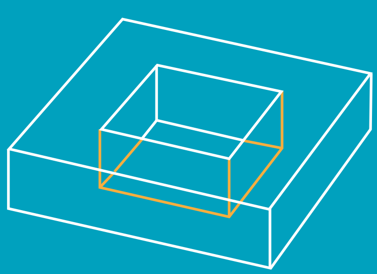




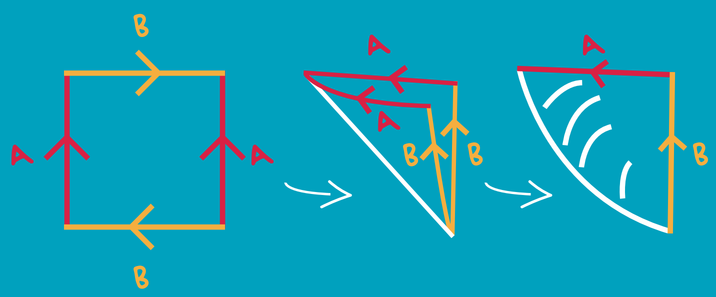
A torus can be represented by a square with labelled sides. Imagine it's made of rubber which can be stretched and squashed. The sides must be pulled around into the third dimension, with the different arrows joining up to make the torus shape:



It wouldn't have to be a smooth shape. It could have square cross-sections rather than circular ones, like this:

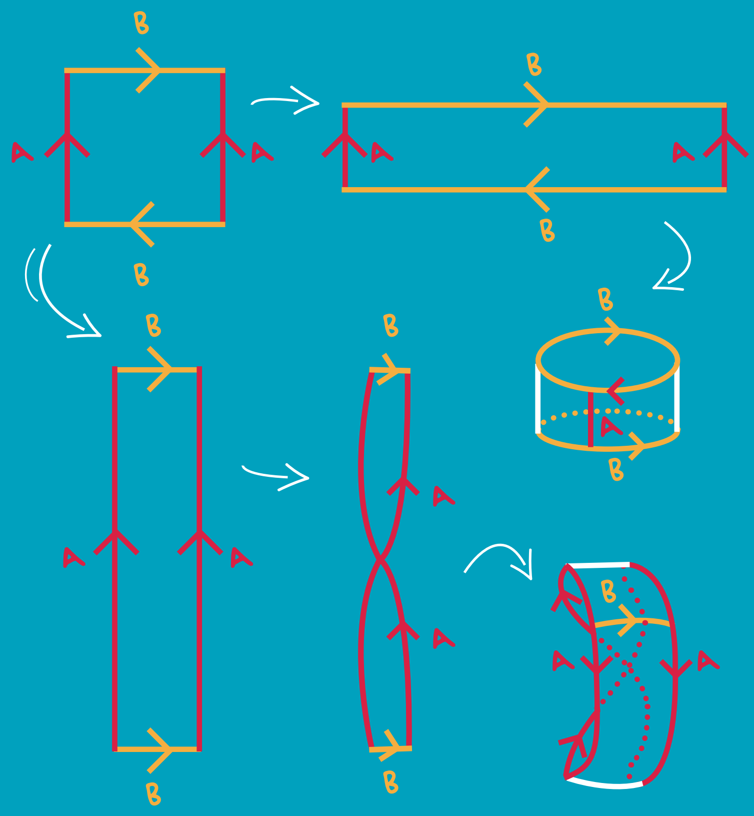


It is possible to represent a sphere by a labelled square too:



You fold it over like an apple turnover. You would need to blow it up a bit to make it look like a conventional sphere, but it has the same properties as a sphere.

So what about other ways of labelling a square? If you label a square as the one below is labelled, it is a bit like a torus.



If you match the "a" sides with arrows agreeing you get a cylinder. But if you want to match the "b" sides you have to introduce a twist to match up the arrows, so you get a Möbius strip. (Notice that in my picture it looks as if the two "a" sides pass through each other because I am drawing in two dimensions, but in three dimensions one would be above the other, giving the rectangle a twist).

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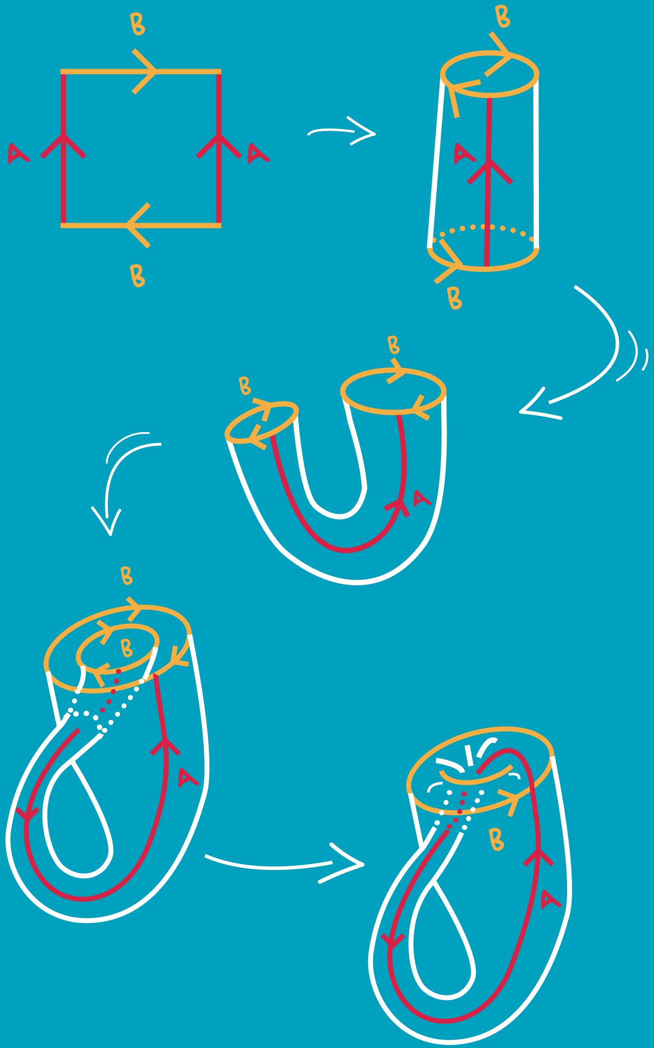
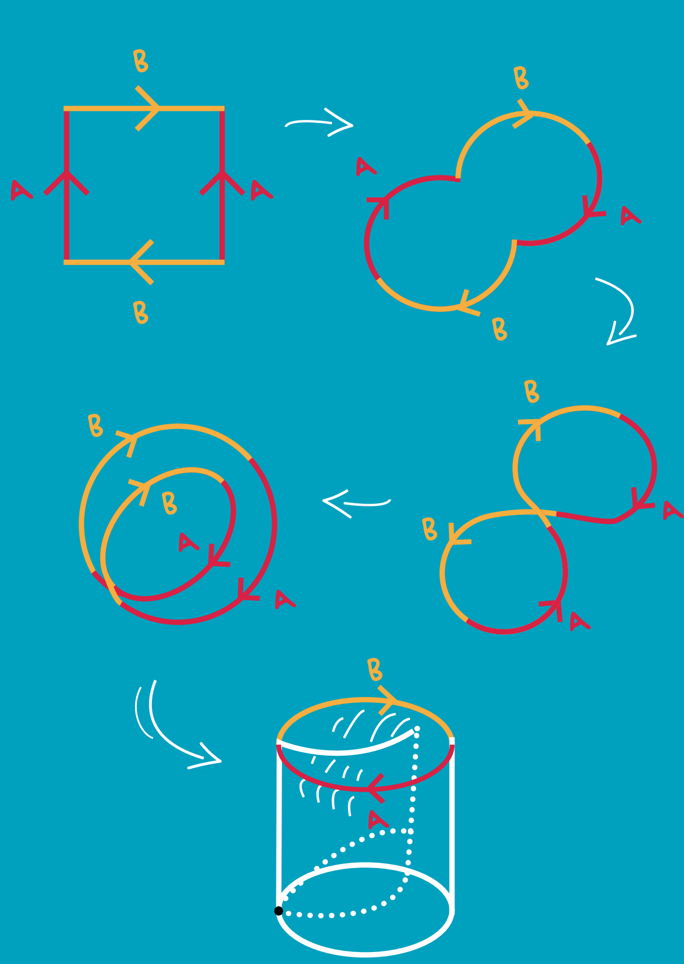
It is impossible to match up both the “a” sides and the “b” sides without passing through the surface itself. In this model we have to make a hole in the surface to push a bit of tube through. If you could fold it up in the fourth dimension, you would be able to do it without passing through the surface, giving you a Klein bottle.



There are lots of pictures of Klein bottles on the web. The one you can make here has rectangular cross-sections, because that is much easier to make in paper. There is a three-dimensional drawing of it in the “3D Klein bottle” GeoGebra file.

There is another way of labelling the square to make a last surface: the real projective plane. This makes a Möbius strip if you join the “a” sides or the “b” sides. There are various ways of folding this up in the fourth dimension. This is my way, but there are other ways you can find on the web. However you do it, if you only have three dimensions you have to pass through the surface. In this model the intersection is a line at the front.



Imagine a surface like a soap bubble between four sides of stretchy wire. Pinch in the two opposite corners, then pick up the bottom left two sides and twist them over to bring the “a” sides and “b” sides closer, and then bend them over the top of the two sides which were originally in the top right. Then imagine pulling all the bits of wire out of the page, leaving a soap bubble trace, and join the “a” sides and “b” sides together to complete the surface.

This is easier to imagine if you make the 3-dimensional model!

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What you'll need

A copy of "Klein bottle net" and one of "Projective plane net"

Two sheets of A4 OHP acetate suitable for printing

Clear sticky tape

Ruler

Scissors

Step-by-Step Instructions

1 Photocopy the nets onto the acetate.

2 For each net, cut along the black solid lines.

3 Discard the small shaded square which falls off of the Klein bottle net.

4 Use the ruler and the back of a scissor blade to score along the dotted lines.

5 Every red dotted line is where the net should be folded up towards you.

6 On the Klein bottle net there are green dotted lines. These should be folded away from you.

6 Both nets are labelled so that you join the edge "a" to edge "a1", "b" to "b1" and so on. Unlabelled edges are obvious as they can only join in one place. Fold the shape together so that labelled edges match and stick edges together with pieces of clear sticky tape.

Quick Tips

The more accurate you are with the cutting out, the better the final surface will look.

Make sure all the dotted lines are scored and all folds are firmly made before sticking any edges together. All folds in the Klein bottle are at 90°. Some of those in the projective plane are less than this. The top of the projective plane bends gently over to match "o" to "o1" and "p" to "p1" so it has not been given a dotted line to make a fold.

Ball point pens which no longer have any ink in them are good for scoring along the dotted lines. They make it easier to fold in a good line but can't cut through.

It is best to fold everything together and make sure you can see where everything goes before you stick anything. Once you have everything in approximately the right place you will be able to see which bits are going to be hard to reach to stick them together. It is best to stick these first.

The Klein bottle and the projective plane can also be printed onto paper to make the surfaces. The acetate is more satisfying because you can see through the sides to see how it is constructed inside.

Once the projective plane is assembled you should be able to follow the thick blue line around with your eyes and see that it is a Möbius strip. Both of these surfaces are "non-orientable". They do not have an inside and an outside. If you start on one side of the acetate and travel around on either of the surfaces, you should be able to see that you can get to the place where you started but be on the other side of the acetate.